RAMAKRISHNA MISSION VIDYAMANDIRA

(Residential Autonomous College under University of Calcutta)

B.A./B.SC. SECOND SEMESTER EXAMINATION, MAY-JUNE 2013

FIRST YEAR

MATHEMATICS (Honours)

Date : 20/5/2013

Time : 11 am – 3 pm

Paper : II

Full Marks : 100

[Use separate Answer Books for each group]

Group – A

Answer **any five** questions from **Q. No. 1-8** and **any five** questions from **Q. No. 9-16** :

1. If z is a variable complex number such that $\left|\frac{z-i}{z+1}\right| = K$, show that the point z lies on a circle on the complex plane if $K \neq 1$ and z lies on a straight line if K = 1. [5]

2. a) Find the principal amplitude of z where $z = 1 + i\tan\theta$, $\pi/2 < \theta < \pi$
   b) Find all complex number z such that exp z = –1. [3+2]

3. State and prove Cauchy-Schwarz inequality. [5]

4. a) If x, y, z are positive real numbers and x + y + z = 1, prove that $(1-x)(1-y)(1-z) \leq \frac{8}{27}$.
   b) Find the maximum value of $(x+2)^5(7-x)^4$ when $-2 < x < 7$. [2+3]

5. a) Find the range of values of r for which the equation $3x^4 + 8x^3 - 6x^2 - 24x + r = 0$ has four real and unequal roots.
   b) Apply Descartes' rule of signs to find the nature of the roots of $x^7 + x^5 - x^3 = 0$ [3+2]

6. Find the equation whose roots are squares of the roots of $x^4 - x^3 + 2x^2 - x + 1 = 0$ and use Descartes' rule of signs to the resulting equation to deduce that the given equation has no real root. [5]

7. Prove that the equation $(x+1)^4 = a(x^4+1)$ is a reciprocal equation if $a \neq 1$ and solve it when a = –2. [5]

8. Solve by Cardan's method : $x^3 + 3x^2 - 3 = 0$ [5]

9. Let $\{u_n\}$ be a monotone decreasing sequence of positive real numbers.
   Prove that $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} 2^n u_{2^n}$ converge or diverge together. [5]

10. Answer either **(a) or (b)** :
    a) Show that the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$ converges to log 2, but the rearranged series $1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} + \frac{1}{3} - \frac{1}{10} - \frac{1}{12} - \frac{1}{14} - \frac{1}{16} + \frac{1}{5} - \ldots$ converges to 0. [5]
    b) State and prove Leibnitz's test for convergence of an alternating series of real numbers. [1+4]

11. a) Exhibit an open cover of the set $\left\{\frac{1}{n} : n \in \mathbb{N}\right\}$ that has no finite subcover.
    b) If K_1 and K_2 are disjoint compact sets, show that there exist $\alpha_i \in K_i$, i = 1,2 such that $0 < |\alpha_1 - \alpha_2| = \inf\{|x_1 - x_2| : x_1 \in K_1, x_2 \in K_2\}$ [2+3]

12. a) Let $c \in \mathbb{R}$ and a function $f : \mathbb{R} \to \mathbb{R}$ is continuous at c. If for every positive δ there is a point y in $(c-\delta, c+\delta)$ such that f(y) = 0, prove that f(c) = 0. [3]
    b) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous on $\mathbb{R}$. Prove that $Z(f) = \{x \in \mathbb{R} \mid f(x) = 0\}$ is a closed set in $\mathbb{R}$. [2]

13. a) A function $f:\mathbb{R}\to\mathbb{R}$ is defined by

$$f(x)=x^2+1, x\in\mathbb{Q}$$
$$=x, x\in\mathbb{R}-\mathbb{Q}$$

Prove that f has a discontinuity of the second kind at every point c in $\mathbb{R}$. [3]

b) A function $f:[0,1]\to\mathbb{R}$ is continuous on [0,1] and f assumes only rational values on [0,1]. Prove that f is a constant. [2]

14. Let f be derivable in a closed and bounded interval [a,b] and $f'(a)\neq f'(b)$. If K is any real number lying between $f'(a)$ and $f'(b)$, show that there exists at least one point $c\in(a,b)$ such that $f'(c)=K$. [5]

15. a) Prove that the function f defined on $\mathbb{R}$ by $f(x)=\dfrac{1}{x^2+1}$, $x\in\mathbb{R}$ is uniformly continuous on $\mathbb{R}$. [2]

b) A function f is differentiable on [0,2] and f(0) = 0, f(1) = 2, f(2) = 1. Prove that $f'(c)=0$ for some $c\in(0,2)$. [3]

16. a) Find a and b in order that $\lim\limits_{x\to 0}\dfrac{a\sin 2x-b\sin 3x}{5x^3}=1$. [2]

b) A line is drawn through a fixed point (a,b) [a > 0, b > 0] to meet the positive direction of the co-ordinate axes at P and Q respectively. Show that the minimum value of PQ is $\left(a^{2/3}+b^{2/3}\right)^{3/2}$. [3]

Group – B

Answer **any four** questions from **Q. No. 17-22** and **any three** questions from **Q. No. 23-27** :

17. a) Determine the matrices A and B, where $A+2B=\begin{bmatrix}1&2&0\\6&-3&3\\-5&3&1\end{bmatrix}$ and $2A-B=\begin{bmatrix}2&-1&5\\2&-1&6\\0&1&2\end{bmatrix}$. [3]

b) Show that a Skew symmetric determent of third order vanishes. [2]

18. Evaluate $\begin{vmatrix}a&b&c\\b&c&a\\c&a&b\end{vmatrix}\begin{vmatrix}p&q&r\\q&r&p\\r&p&q\end{vmatrix}$ and hence express $(a^3+b^3+c^3-3abc)(p^3+q^3+r^3-3pqr)$ in the form $l^3+m^3+n^3-3lmn$. [5]

19. Use elementary row operations on A to obtain A^{-1} where A is $\begin{pmatrix}2&0&0\\4&3&0\\6&4&1\end{pmatrix}$. [5]

20. Prove that the matrix $\dfrac{1}{3}\begin{bmatrix}1&-2&2\\2&-1&-2\\2&2&1\end{bmatrix}$ is orthogonal. [2]

Utilise this to solve the equations : $\begin{aligned}x-2y+2z&=2\\2x-y-2z&=1\\2x+2y+z&=7\end{aligned}$. [3]

21. Let $S=\{(x,y,z)\in\mathbb{R}^3:3x-y+z=0,3x-y-z=0\}$. Show that S is a subspace of $\mathbb{R}^3$. Find a basis of S. Find also the Dimension of the Subspace. [1+2+2]

22. Find a basis of the subspace $S\cap T$ of $\mathbb{R}^4$, where $S=\{(x,y,z,w)\in\mathbb{R}^4:x+y+z+w=0\}$ and $T=\{(x,y,z,w)\in\mathbb{R}^4:2x+y-z+w=0\}$ [5]

23. a) Solve graphically the following Linear programming problems :

Maximize $z = 5x_1 + 7x_2$

subject to $3x_1 + 8x_2 \leq 12$

$2x_1 \leq 3$

and $x_1, x_2 \geq 0$ [2]

b) Given that $x_1 = 1$, $x_2 = 3$, $x_3 = 2$ is F.S. of the equations

$2x_1 + 4x_2 - 2x_3 = 10$

$10x_1 + 3x_2 + 7x_3 = 33$

Reduce the above F.S. to a B.F.S. by reduction theorem. [3]

c) At a cattle treading firm, it is prescribed that the food ration for one animal must contain atleast 14, 22 and 11 units of nutrients A, B and C respectively. Two different kinds of the fodder are available. Each unit weight of these two contains the following amounts of three nutrients.

	Fodder – 1	Fodder – 2
Nutrient A	2	1
Nutrient B	2	3
Nutrient C	1	1

It is given that the cost of fodder 1 and 2 are 3 and 2 monetary units respectively. Formulate the problem of finding the minimum cost of purchasing the fodders as a L.P.P. [5]

24. a) Prove that the dual of the dual is the primal. [4]

b) A salesman has to visit five cities A, B, C, D and E. The distances (in hundred miles) between the five cities are as follows :

From City		A	B	C	D	E
	A	∞	14	10	24	41
	B	6	∞	10	12	10
	C	7	13	∞	8	15
	D	11	14	30	∞	17
	E	6	8	12	16	∞

If the salesman starts from the city A and has to come back at city A, which route should he select so that the total distance travelled is minimum? [6]

25. a) Show that every extreme point of the convex set of all feasible solutions of the set of equations $Ax = b$, $x \geq 0$ corresponds to a B.F.S. [5]

b) Reduce the feasible solution (2, 1, 1) of the system

$x_1 + 4x_2 - x_3 = 5$

$2x_1 + 3x_2 + x_3 = 8$, $x_1, x_2, x_3 \geq 0$ to a basic feasible solution. [2]

c) Find all the basic solutions of

$4x_1 + 2x_2 + 3x_3 - 8x_4 = 6$

$3x_1 + 5x_2 + 4x_3 - 6x_4 = 8$

determine which of them are feasible also. [3]

26. a) Solve by simplex method :

Maximize $z = 2x_1 + 3x_2$

subject to $x_1 + x_2 \leq 8$

$x_1 + 2x_2 = 5$

$2x_1 + x_2 \leq 8$

$x_1 \geq 0,\ x_2 \geq 0$ [6]

b) Find the optimal assignment and the corresponding assignment cost from the following cost matrix [4]

	A	B	C	D	E
1	9	8	7	6	4
2	5	7	5	6	8
3	8	7	6	3	5
4	8	5	4	9	3
5	6	7	6	8	5

27. a) Write down the dual of the following problem :

Minimize z = $30x_1 + 36x_2$

subject to $x_1 + x_2 \geq 5$

$2x_1 + 3x_2 \geq 2$

$-2x_1 + x_2 \geq 2$

$x_1, x_2 \geq 0$

Solving the dual problem find out the optimal solution and the optimal value of the objective function of the primal. [5]

b) Solve the following transportation problem : [5]

	A	B	C	a_i
I	6	8	4	14
II	4	9	3	12
III	1	2	6	5
b_j	6	10	15	